Pearson Edexcel Level 3	
GCE Mathematics **Advanced** **Paper 2: Pure Mathematics**	**PMT Mock 2**
Time: 2 hours	**Paper Reference(s)** **9MA0/02**
You must have: **Mathematical Formulae and Statistical Tables, calculator**	

Candidates may use any calculator permitted by Pearson regulations. Calculators must not have the facility for algebraic manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use black ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need*.
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 14 questions in this paper. The total mark is 100.
- The marks for each question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

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Answer ALL questions.

1. The figure 1 shows part of the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \frac{ax+4}{x-b}$, $x > 2$

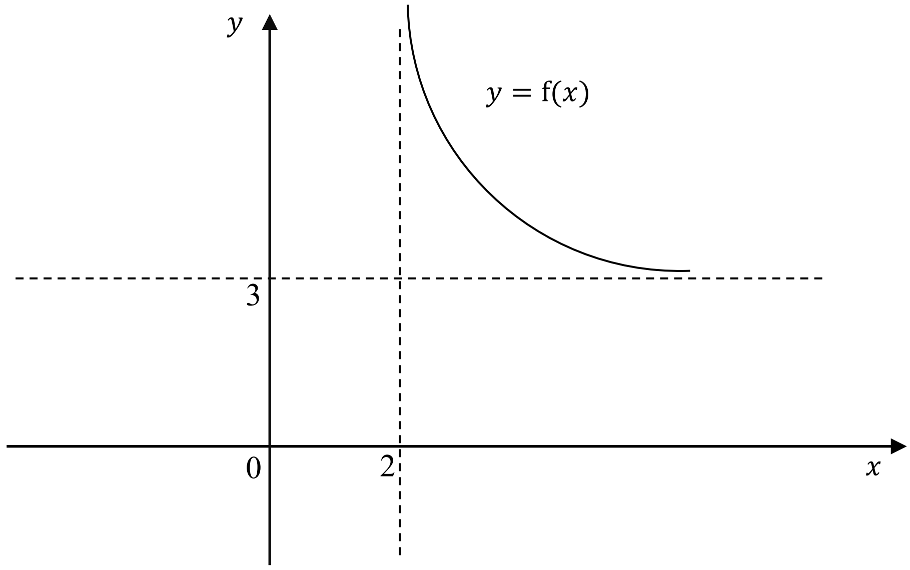


Figure 1

a. State the values of a and b.

B1 $b = 2$ B1 $a = 3$

(2)

b. State the range of f.

B1 $\mathrm{f}(x) > 3$ or $y > 3$

(1)

c. Find $\mathrm{f}^{-1}(x)$, stating its domain.

$y = \frac{2x+4}{x-2}$

$x = \frac{2y+4}{y-2}$

$x(y-2) = 3y - 4$

$xy - 2x = 3y - 4$

$xy - 3y = 2x - 4$

$y(x-3) = 2x - 4$

$y = \frac{2x-4}{x-3}, x > 3$

M1 Attempts at the method for finding the inverse

Look for a minimum of cross multiplying by $(x-2)$ and proceeding to form a

A1 Correct inverse $\mathrm{f}^{-1}(x) = \frac{2x+4}{x-3}$, or $y = \frac{2x+4}{x-3}$

A1 Correct domain $x > 3$

(3)

(Total for Question 1 is 6 marks)

2. Relative to a fixed origin O,

the point A has position vector $(3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

the point B has position vector $(\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$

and the point C has position vector $(-\mathbf{i} + \mathbf{j} + a\mathbf{k})$, where a is a constant and $a > 0$.

Given that $|\overrightarrow{BC}| = \sqrt{41}$

a. show that $a = 2$.

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} \quad \Rightarrow \quad \overrightarrow{BC} = (-\mathbf{i} + \mathbf{j} + a\mathbf{k}) - (\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$$

$$\overrightarrow{BC} = -2\mathbf{i} - \mathbf{j} + (a + 4)\mathbf{k}$$

$$(-2)^2 + (-1)^2 + (a + 4)^2 = 41$$

$$(a + 4)^2 = 36$$

$$a = 2 \quad \text{or} \quad a = -10$$

From the question, $a > 0$, so we reject the solution $a = -10$ and have $a = 2$

M1 Finds the difference between $\overrightarrow{OC}$ and $\overrightarrow{OB}$ then squares and adds each of the 3 components

M1 Complete method of correctly applying Pythagoras Theorem on $|\overrightarrow{BC}| = \sqrt{41}$ and using a correct method of solving their resulting quadratic equation to find at least one of $a = \ldots$

A1 Obtains only one value, $a = 2$, $a>0$

(3)

D is the point such that $ABCD$ forms a parallelogram.

b. Find the position vector of D.

$\overrightarrow{OD} = \overrightarrow{AD} + \overrightarrow{OA}$

$\overrightarrow{BC} = \overrightarrow{AD} = -2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$-2\mathbf{i} - \mathbf{j} + 6\mathbf{k} = \overrightarrow{OD} - (3\mathbf{i} - \mathbf{j} + 2\mathbf{k})$

$\overrightarrow{OD} = \mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$

M1 Complete applied strategy to find a vector expression for $\overrightarrow{OD}$

A1 Correct position vector for $\overrightarrow{OD} = \mathbf{i} - 2\mathbf{j} + 8\mathbf{k}$

(2)

(Total for Question 2 is 5 marks)

3. a. "If p and q are irrational numbers, where $p \neq q, q \neq 0$, then $\frac{p}{q}$ is also irrational."

Disprove this statement by means of a counter example.

M1 States or uses any pair of different numbers that will disprove the statement.

e.g. $\sqrt{2}, \sqrt{8}$; $\sqrt{3}, \sqrt{12}$; $\pi, 2\pi$; $3e, 7e$

e.g. $p = \sqrt{2}$, $q = \sqrt{8}$ $\quad \frac{p}{q} = \frac{\sqrt{2}}{\sqrt{8}} \Rightarrow \frac{\sqrt{2}}{\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}} = \frac{1}{2}$

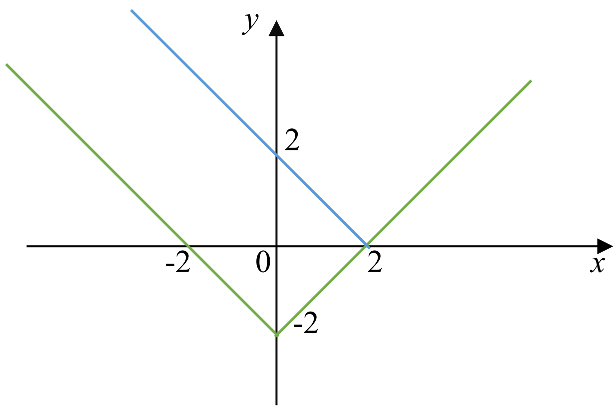
$\Rightarrow$ statement untrue or $\frac{1}{2}$ is not irrational or $\frac{1}{2}$ is rational

A1 Uses correct reasoning to disprove the given statement, with a correct conclusion

(2)

b. (i) Sketch the graph of $y = |x| - 2$.

(ii) Explain why $|x - 2| \geq |x| - 2$ for all real values of x.



B1 V shaped graph symmetrical about the y-axis, with intercepts (2,0), (0,-2) and (2,0) (see green graph)

M1 draws the graph of $y = |x - 2|$ on top of the graph of $y = |x| - 2$ (see blue graph)

A1 the graph of $y = |x - 2|$ is either the same or above the graph of $y = |x| - 2$

or when $x \geq 0$, the graph of $y = |x - 2|$ is above the graph of $y = |x| - 2$

(3)

(Total for Question 3 is 5 marks)

4. (a) Show that $\sum_{r=1}^{20}(2^{r-1}-3-4r)=1047675$

$$\sum_{r=1}^{20}(2^{r-1}-3-4r)=\sum_{r=1}^{20}2^{r-1}-\sum_{r=1}^{20}(3+4r)$$

$$=2^0+2^1+2^2+\cdots+2^{19}-(7+10+13+\cdots+83)$$

The sequence $2^0+2^1+2^2+\cdots+2^{19}$ is a geometric progression with $a=1, r=2$. The sum of the sequence is given by:

$$S_n=\frac{a(r^n-1)}{r-1}$$

$$S_{20}=\frac{1(2^{20}-1)}{2-1}=1048575$$

The sequence $7+10+13+\cdots+83$ is an arithmetic progression with $a=7, d=3$. The sum of the sequence is given by:

$$S_n=\frac{1}{2}n(a+l)$$

$$S_{20}=10(7+83)=900$$

Therefore:

$$\sum_{r=1}^{20}(2^{r-1}-3-4r)=1048575-900=1047675$$

M1 uses a correct method to find the given sum

M1 Correct method for finding the sum of a geometric progression

M1 Correct method for finding the sum of an arithmetic progression

A1 Using correct formula and showing all steps fully and leading to 1047675

(4)

(b) A sequence has nth term $u_n = \sin(90n^0)$, $n \geq 1$

(i) Find the order of the sequence.

$u_1 = \sin(90^0) = 1,$ $u_2 = \sin(180^0) = 0,$

$u_3 = \sin(270^0) = -1,$ $u_4 = \sin(360^0) = 0,$ $u_5 = \sin(450^0) = 1, \ldots$

Order 4

B1 obtains the terms of this sequence are 1, 0, -1, 0, 1, 0, … and deduces the order 4

(1)

(ii) Find $\sum_{r=1}^{222} u_r$

$\frac{222}{4} = 55.5$, so we need 55 and a half 'cycles' of the sequence

$$55(1 + 0 + \ -1 + 0) + 1 + 0 = 1$$

M1 for deducing that the sum can be found using the order of the sequence

A1 obtains 1

(2)

(Total for Question 4 is 7 marks)

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5. $\mathrm{f}(x) = \frac{1}{3}x^3 - 4x - 2$

a. Show that the equation $\mathrm{f}(x) = 0$ can be written in the form $x = \pm\sqrt{a + \frac{b}{x}}$, and state the values of the integers a and b.

$$\frac{1}{3}x^3 - 4x - 2 = 0 \Rightarrow x^3 = 12x + 6 \ \Rightarrow x^2 = 12 + \frac{6}{x} \Rightarrow x = \pm\sqrt{12 + \frac{6}{x}}$$

M1 equates $\mathrm{f}(x) = 0$ and proceeds to an equation in $x = \pm\sqrt{\ldots}$

A1 correct values of $a = 12$, $b = 6$

(2)

$\mathrm{f}(x) = 0$ has one positive root, α.

The iterative formula $x_{n+1} = \sqrt{a + \frac{b}{x_n}}$, $x_0 = 4$ is used to find an approximation value for α .

b. Calculate the values of x_1, x_2, x_3 and x_4 to 4 decimal place

$$x_1 = \sqrt{12 + \frac{6}{4}} = 3.6742\,, \quad x_2 = 3.6923\,, \quad x_3 = 3.6912\,, \quad x_4 = 3.6913$$

M1 Uses a formula $x = \sqrt{a + \frac{b}{x}}$ with 4 to find the value of x_1 correct to 4 d.p.

A1 correct values for $x_1 = 3.6742$, $x_2 = 3.6923$, $x_3 = 3.6912$, $x_4 = 3.6913$

(2)

c. Explain why for this question, the Newton-Raphson method cannot be used with $x_1 = 2$.

B1 Accepts any reasonable reason why the Newton-Raphson method cannot be used with $x_1 = 2$

e.g. There is a stationary point at $x = 2$

Tangent to the curve would not meet the x-axis

Tangent to the curve is horizontal

(1)

(Total for Question 5 is 5 marks)

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6. $f(x) = 2x^3 + 3x^2 - 1$

a. (i) Show that $(2x - 1)$ is a factor of $f(x)$.

(ii) Express $f(x)$ in the form $(2x - 1)(x + a)^2$ where a is an integer.

(i) If $(2x - 1)$ is a factor then by the factor theorem, $f\left(\frac{1}{2}\right) = 0$

$\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 1 = 0$ and so $(2x - 1)$ is a factor.

B1 Correctly applies the factor theorem

(ii) $(2x - 1)(x^2 + 2ax + a^2) = 2x^3 + 3x^2 - 1$

$(2x^3 + 4ax^2 + 2a^2x - x^2 - 2ax - a^2) = 2x^3 + 3x^2 - 1$

$2x^3 + (4a - 1)x^2 + (2a^2 - 2a)x - a^2 = 2x^3 + 3x^2 - 1$

$$(4a - 1) = 3 \Rightarrow a = 1, \quad (2a^2 - 2a) \Rightarrow a = 1$$

M1 Attempts to find the other quadratic factor $(x^2 \pm ax \pm b)$ by long division or equating coefficients

M1 factorises 3 terms quadratic to obtain $(x \pm a)^2$ e.g. $(x + 1)^2$

A1 correct answer only $(2x - 1)(x + 1)^2$

(4)

Using the answer to part a) (ii)

b. show that the equation $2p^6 + 3p^4 - 1$ has exactly two real solutions and state the values of these roots.

Let $x = p^2$, therefore the equation becomes

$$2x^3 + 3x^2 - 1 = 0$$

Using part ii), this factorises into $(2x - 1)(x + 1)^2 = 0 \Rightarrow x = \frac{1}{2}, x = -1$ (repeated root).

$$x = \frac{1}{2} \Rightarrow p^2 = \frac{1}{2} \Rightarrow p = \pm\frac{1}{\sqrt{2}}$$

$$x = -1 \Rightarrow p^2 = -1 \Rightarrow \text{no real roots}$$

So there are two real roots

M1 Uses part ii) to factorise and give a partial explanation that $p^2 = -1$ has no real roots

A1 Complete proof that the given equation has exactly two real solutions

(2)

c. deduce the number of real solutions, for $5\pi \leq \theta \leq 8\pi$, to the equation

$$2\cos^3\theta + 3\cos^2\theta - 1 = 0$$

Let $y = \cos\theta \Rightarrow \cos\theta = -1, \frac{1}{2}$.

For $5\pi \leq \theta \leq 8\pi$, $\cos\theta = -1 \Rightarrow \theta = 5\pi, 7\pi$

$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{17}{3}\pi, \frac{19}{3}\pi, \frac{23}{3}\pi$

Therefore, there are 5 solutions

B1 deduces that there are 5 solutions

(1)

(Total for Question 6 is 7 marks)

7. i. Solve $0 \leq \theta \leq 180^0$, the equation

$$4\cos\theta = \sqrt{3}\ \text{cosec}\,\theta$$

$$4\cos\theta = \frac{\sqrt{3}}{\sin\theta}$$

$$4\cos\theta\sin\theta = \sqrt{3}$$

Using $\sin 2\theta = 2\sin\theta\cos\theta$

$$2\sin 2\theta = \sqrt{3} \Rightarrow \sin 2\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{1}{2}\arcsin\left(\frac{\sqrt{3}}{2}\right) \Rightarrow \theta = 30°, 60°$$

B1 for $\text{cosec}\,\theta = \frac{1}{\sin\theta}$

M1 Applying $\sin 2\theta = 2\sin\theta\cos\theta$ and proceeding to $\sin 2\theta = k$,

M1 Use the correct order of operations to find at least one value for θ in ether radians or degrees

A1 both values are correct and no extra values in range.

(4)

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(ii) Solve, for $0 \le x \le 2\pi$, the equation

$$\cos x - \sqrt{3}\sin x = \sqrt{3}$$

$\cos x - \sqrt{3}\sin x = \sqrt{3} \Rightarrow R\cos(x - \alpha) = \sqrt{3}$

$R^2 = 1^2 + \left(-\sqrt{3}\right)^2 = 4 \Rightarrow R = 2$

$\tan\alpha = \frac{-\sqrt{3}}{1} \Rightarrow \alpha = -\frac{\pi}{3}$

$2\cos\left(x + \frac{\pi}{3}\right) = \sqrt{3}$

$x = (\cos^{-1}\frac{\sqrt{3}}{2}) - \frac{\pi}{3}$

$x = \frac{3\pi}{2}, \frac{11\pi}{6}$

M1 expresses $\cos x - \sqrt{3}\sin x = \sqrt{3}$ in the form $R\cos(x \pm \alpha) = \sqrt{3}$

M1 uses $R\cos(x + \alpha)$ to find the values of both R and α

A1 $\cos\left(x + \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

M1 Use the correct order of operations to find at least one value for x in either radians or degrees $x = \arccos(\frac{\sqrt{3}}{2}) - \frac{\pi}{3}$

A1 correct solutions $x = \frac{3\pi}{2}, \frac{11\pi}{6}$

(5)

(Total for Question 7 is 9 marks)

8.

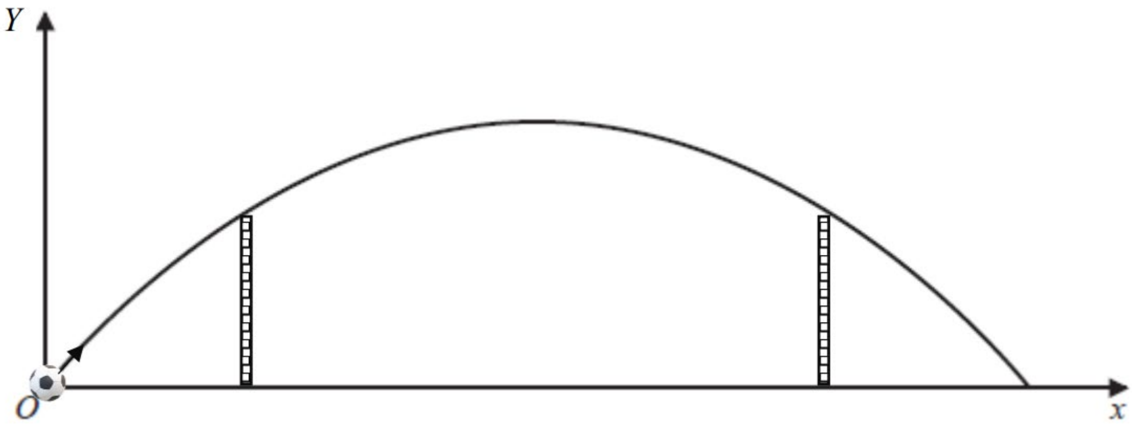

Figure 2

In a competition, competitors are going to kick a ball over the barrier walls. The height of the barrier walls are each 9 metres high and 50 cm wide and stand on horizontal ground.

The figure 2 is a graph showing the motion of a ball.

The ball reaches a maximum height of 12 metres and hits the ground at a point 80 metres from where its kicked.

a. Find a quadratic equation linking Y with x that models this situation.

The maximum is at (40,12) (due to symmetry) so using completing the square:

$$y = a(x-40)^2 + 12$$

Substituting in (0,0):

$$0 = a(-40)^2 + 12$$

$$-1600a = 12 \Rightarrow a = \frac{-3}{400}$$

$$y = \frac{-3}{400}(x-40)^2 + 12$$

M1 translates the situation given into a suitable equation for the model.

M1 Applies a complete strategy with appropriate constraints to find all constants in their model.

A1 Finds a correct equation linking Y to x

$$Y = -\frac{3}{400}(x-40)^2 + 12\,, \quad Y = -\frac{3}{400}x(x-80)$$

(3)

The ball passes over the barrier walls.

b. Use your equation to deduce that the ball should be placed about 20 m from the first barrier wall.

$$9 = \frac{-3}{400}(x-40)^2 + 12$$

$$(x-40)^2 = 400$$

$$x = 20, x = 60$$

So the ball should be placed 20m from the first barrier wall.

M1 Substitutes $Y = 9$ into their quadratic equation and proceeds to obtain a term quadratic equation or a quadratic in the form $(x \pm c)^2 = d$

M1 Correct method of solving their quadratic equation to give at least one solution

A1 chooses the correct value for x i.e. 20

(3)

c. Give one limitation of the model.

B1 Gives a limitation of the model

- The ground is horizontal
- The ball needs to be kicked from the ground
- The ball is modelled as a particle
- There is no wind or air resistance on the ball
- There is no spin on the ball
- The trajectory of the ball is a perfect parabola

(1)

(Total for Question 8 is 7 marks)

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9. Given that x is measured in radians, prove, from the first principles, that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$$

You may assume the formula for $\sin(A \pm B)$ and that as $h \to 0, \frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h-1)+\sin h\cos x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h-1)}{h} + \lim_{h\to 0}\frac{\sin h\cos x}{h}$$

$$= \sin x\ \lim_{h\to 0}\frac{\cos h-1}{h} + \cos x\lim_{h\to 0}\frac{\sin h}{h}$$

Using the limits provided in the question:

$$= \cos x$$

B1 Gives the correct fraction such as $\frac{\sin(x+h)-\sin x}{h}$

M1 uses the compound angle formula for $\sin(x+h)$ to give $\sin x\ \cos h \pm \cos x\ \sin h$

A1 Achieves $\frac{\sin x\cos h+\cos x\sin h}{h}$

M1 Complete attempt to apply the given limits to the gradient of their chord

A1 Correct solution only $\frac{\mathrm{d}}{\mathrm{d}x} = (\sin x) = \cos x$

(5)

(Total for Question 9 is 5 marks)

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10. Given that $y = 8$ at $x = 1$, solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(12x+9)y^{\frac{1}{3}}}{x}$$

Giving your answer in the form $y^2 = \mathrm{f}(x)$.

Using separation of variables:

$$\int \frac{1}{y^{\frac{1}{3}}}\,\mathrm{d}y = \int \frac{(12x+9)}{x}\,\mathrm{d}x$$

$$\frac{3}{2}y^{\frac{2}{3}} = 12x + 9\ln x + c$$

Substituting in $y = 8, x = 1$:

$$\frac{3}{2}(8)^{\frac{2}{3}} = 12(1) + 9\ln(1) + (c)$$

$$c = -6$$

So,

$$y^2 = (8x + 6\ln x - 4)^3$$

B1 Separates variables correctly. No need for integral signs

M1 Integrates left hand side $Ay^{-\frac{1}{3}}$ to give $By^{\frac{2}{3}}$

M1 Integrates the right hand side $A + \frac{B}{x}$ and obtains $Cx + D\ln x + (c)$

A1 Correct answer for lhs and rhs

M1 Substitutes $y = 8$ at $x = 1$ into their function and finds the value of the constant c and attempts to find $y^2 = \cdots$.

A1 Correct solution only

(Total for Question 10 is 6 marks)

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11. $\frac{-6x^2+24x-9}{(x-2)(1-3x)} \equiv A + \frac{B}{x-2} + \frac{C}{1-3x}$

a. Find the values of the constants A, B and C.

$$\frac{-6x^2 + 24x - 9}{(x-2)(1-3x)} \equiv A + \frac{B}{x-2} + \frac{C}{1-3x}$$

$$-6x^2 + 24x - 9 = A(x-2)(1-3x) + B(1-3x) + C(x-2)$$

Substituting $x = 2$:

$$-6(2)^2 + 24(2) - 9 \equiv A(2-2)\big(1-3(2)\big) + B\big(1-3(2)\big) + C\big((2)-2\big)$$

$$15 = -5B \Rightarrow B = -3$$

Substituting $x = \frac{1}{3}$

$$-6\left(\frac{1}{3}\right)^2 + 24(\frac{1}{3}) - 9 \equiv A\left(\frac{1}{3} - 2\right)\left(1 - 3\left(\frac{1}{3}\right)\right) + B\left(1 - 3\left(\frac{1}{3}\right)\right) + C((\frac{1}{3}) - 2)$$

$$-\frac{5}{3} = -\frac{5}{3}C \Rightarrow C = 1$$

Substituting $x = 1$ (can be any value)

$$-6(1)^2 + 24(1) - 9 = A(1-2)(1-3) - 3(1-3) + (1-2)$$

$$9 = 2A + 5 \Rightarrow A = 2$$

M1 Uses a correct identity

Either $\quad -6x^2 + 24x - 9 \equiv A(x-2)(1-3x) + B(1-3x) + C(x-2)$

OR $\quad 10x - 5 \equiv B(1-3x) + C(x-2)$

in a complete method to find values for B and C.

B1 $\quad A = 2$

M1 Attempts to find the value of either B or C from their identity.

This can be achieved by either substituting values into their identity or by comparing coefficients and solving the resulting equations simultaneously

A1 $\quad B = -3$ and $C = 1$

(4)

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b. Using part (a), find $f'(x)$.

$$f(x) = 2 + \frac{-3}{x-2} + \frac{1}{1-3x}$$
$$f'(x) = \frac{3}{(x-2)^2} + \frac{3}{(1-3x)^2}$$

M1 Differentiates $2 - 3(x-2)^{-1} + (1-3x)^{-1}$ to give $A(x-2)^{-1} + B(1-3x)^{-1}$

A1 correct answer only $3(x-2)^{-2} + 3(1-3x)^{-2}$

(2)

c. Prove that $f(x)$ is an increasing function.

Both terms of $f'(x)$ are positive for all values of x, so $f(x)$ is an increasing function

A1 $f'(x) = +ve \ + +ve > 0$, so $f(x)$ is an increasing function

(1)

(Total for Question 11 is 7 marks)

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12. a. Prove that

$$\frac{\sec^2 x - 1}{\sec^2 x} \equiv \sin^2 x$$

M1 either splits $\frac{\sec^2 x-1}{\sec^2 x} = \frac{\sec^2 x}{\sec^2 x} - \frac{1}{\sec^2 x}$ and uses $\frac{1}{\sec^2 x} = \cos^2 x$

or states or uses $\sec^2 x - 1 = \tan^2 x$ and $\sec^2 x = \frac{1}{\cos^2 x}$

M1 either replaces $1 - \cos^2 x = \sin^2 x$

or replaces $\frac{\tan^2 x}{\frac{1}{\cos^2 x}} = \frac{\sin^2 x}{\cos^2 x} \times \frac{\cos^2 x}{1}$

A1 correct proof showing all necessary intermediate steps with no errors

(3)

b. Hence solve, for $-360^0 < x < 360^0$, the equation

$$\frac{\sec^2 x - 1}{\sec^2 x} = \frac{\cos 2x}{2}$$

Using part i) and the identity $\cos 2x = 1 - 2\sin^2 x$:

$$\sin^2 x = \frac{1 - 2\sin^2 x}{2}$$

$$4\sin^2 x = 1$$

$$\sin x = \pm\frac{1}{2}$$

$$x = 30^0, 150^0, -210^0, -330^0$$

M1 Uses $\frac{\sec^2 x-1}{\sec^2 x} = \sin^2 x$ and replaces $\cos 2x = 1 - 2\sin^2 x$ and obtains an equation of the form $A\sin^2 x = 1$

A1 Correct equation $4\sin^2 x = 1$

A1 For two of $x = 30^0, 150^0, -210^0, -330^0$

A1 For $x = 30^0, 150^0, -210^0, -330^0$ with no additional values in the range

(4)

(Total for Question 12 is 7 marks)

13. a. Find $\int \ln x \ \ \mathrm{d}x$

Using integration by parts:

$$\int 1 \times \ln x = x \ln x - \int x.\frac{1}{x}\ \mathrm{d}x$$

$$= x\ \ln x - \int dx$$

$$= x \ln x - x + c$$

M1 Uses integration by parts the correct way around

A1 Correct expression $x \ln x - \int x.\frac{1}{x}\ \mathrm{d}x$

M1 Integrates the second term $\int 1 \ \ \mathrm{d}x = Dx \qquad D \neq 0$

A1 correct integration with $+\,c \qquad x \ln x - x + c$

(4)

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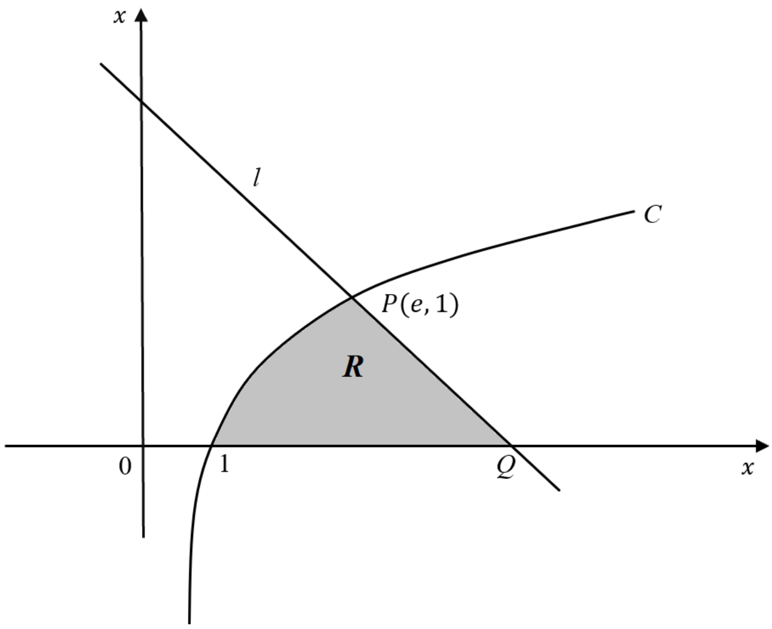

Figure 3

Figure 3 shows a sketch of part of the curve with equation

$$y = \ln x, \quad x > 0$$

The point P lies on C and has coordinate $(e, 1)$.

The line l is a normal to C at P. The line l cuts the x-axis at the point Q.

b.Find the exact value of the x-coordinate of Q.

$$\frac{dy}{dx} = \frac{1}{x}$$

Gradient of C at $(e, 1) = \frac{1}{e}, \Rightarrow$ Gradient of $l = -e$

$$l: y = -ex + c$$

Substituting in $(e, 1)$:

$$1 = -e(e) + c \Rightarrow c = 1 + e^2$$

Substituting $y = 0$ to find x-intercept:

$$0 = -ex + 1 + e^2 \Rightarrow x = \frac{e^2 + 1}{e}$$

B1 differentiates $\ln x$ to give $\frac{1}{x}$

M1 Complete strategy to find the x-coordinate where their normal to C at $P(e,1)$ meets the x-axis

A1 l meets x-axis at $x = \frac{e^2+1}{e}$ or $e + \frac{1}{e}$

(3)

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The finite region **R**, shown shaded in figure 3, is bounded by the curve, the line l and the x-axis.

c. Find the exact area of **R**.

$R = \int_1^e \ln x + \text{triangle } PQ(e, 0)$

$R = \int_1^e \ln x + \frac{\frac{1}{e} \times 1}{2}$

$= (e \ln e - e) - (1 \ln 1 - 1) + \frac{\frac{1}{e} \times 1}{2}$

$= 1 + \frac{1}{2e}$

M1 Complete strategy of finding the area ***R*** by finding the sum of two key areas.

M1 Some evidence of applying limits of e and 1 and subtracts the correct way round

A1 correct solution only $1 + \frac{1}{2e}$

(3)

(Total for Question 13 is 10 marks)

14. A population of ants being studied on an island. The number of ants, P, in the population, is modelled by the equation.

$$P = \frac{900ke^{0.2t}}{1+ke^{0.2t}}, \text{ where } k \text{ is a constant.}$$

Given that there were 360 ants when the study started,

a. show that $k = \frac{2}{3}$.

At time $t = 0$, $P = 360 \Rightarrow 360 = \frac{900k}{1+k}$

$360(1+k) = 900k$

$360 + 360k = 900k \Rightarrow \frac{2}{3}$

M1 setting $P = 360$ at $t = 0$ $\quad \Rightarrow 360 = \frac{900k}{1+k}$

M1 obtains an equation in the form $360 = 540k$

A1 correct solution only $k = \frac{2}{3}$

(3)

b. Show that $P = \frac{1800}{2+3e^{-0.2t}}$.

$$P = \frac{900 \times \frac{2}{3}e^{0.2t}}{1+\frac{2}{3}e^{0.2t}} = \frac{1800e^{0.2t}}{3+2e^{0.2t}} = \frac{1800}{3e^{-0.2t}+2}$$

B1 substitutes $k = \frac{2}{3}$ and divides both numerator and denominator by $e^{0.2t}$

i.e. $P = \frac{1800}{2+3e^{-0.2t}}$

(1)

The model predicts an upper limit to the number of ants on the island.

c. State the value of this limit.

B1 sets $t \to \infty$ $\quad P = 900$

(1)

d. Find the value of t when $P = 520$. Give your answer to one decimal place.

$$520 = \frac{1800}{2 + 3e^{-0.2t}}$$

$$2 + 3e^{-0.2t} = \frac{1800}{520}$$

$$3e^{-0.2t} = \frac{19}{13}$$

$$e^{-0.2t} = \frac{19}{39}$$

$$-0.2t = \ln\frac{19}{39}$$

$$t = 3.5956 \ldots$$

M1 substitutes $P = 520$ and proceeds to an equation of the form $Ae^{-0.2t} = B$

A1 correct equation $1560e^{-0.2t} = 760$ or equivalent

M1 correct order of operations using ln's to make t the subject.

A1 correct answer only $t = 3.6$

(4)

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e. i. Show that the rate of growth, $\frac{dP}{dt} = \frac{P(900-P)}{4500}$

ii. Hence state the value of P at which the rate of growth is a maximum.

i. $P = \frac{1800}{2+3e^{-0.2t}}$

Using the quotient rule:

$$\frac{dP}{dt} = \frac{1800e^{-0.2t}}{(2+3e^{-0.2t})^2}$$

$$P = \frac{1800}{2+3e^{-0.2t}} \Rightarrow 2+3e^{-0.2t} = \frac{1800}{P} \Rightarrow e^{-0.2t} = \frac{\frac{1800}{P}-2}{3}$$

$$\frac{dP}{dt} = \frac{1800\left(\frac{\frac{1800}{P}-2}{3}\right)}{(\frac{1800}{P})^2}\frac{1800P-2P^2}{9000} = \frac{P(900-P)}{4500}$$

M1 attempts to differentiate using

A1 A correct differentiation statement

i.e. $1080e^{-0.2t}(2+3e^{-0.2t})^{-2}$ or $\frac{1080e^{-0.2t}}{(2+3e^{-0.2t})^2}$

M1 substitutes $2+3e^{-0.2t} = \frac{1800}{P}$ and $e^{-0.2t} = \frac{1800}{P}$ or substitutes $e^{-0.2t} = \frac{\frac{1800}{P}-2}{3}$

into their $\frac{dP}{dt} = \cdots$ to form an equation linking $\frac{dP}{dt}$ and P

A1 Correct algebra leading to $\frac{dP}{dt} = \frac{P(900-P)}{4500}$

ii. $\frac{d^2P}{dt^2} = 900 - 2P$

Rate of growth is at a maximum when $\frac{d^2P}{dt^2} = 0 \Rightarrow P = 450$

B1 450

(5)

(Total for Question 14 is 14 marks)

cc BY NC ND